

A SUBGROUP THEOREM FOR HOMOLOGICAL FILLING FUNCTIONS

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ABSTRACT. We use algebraic techniques to study homological filling functions of groups and their subgroups. If G is a group admitting a finite $(n+1)$ -dimensional $K(G, 1)$ and $H \leq G$ is of type F_{n+1} , then the n^{th} -homological filling function of H is bounded above by that of G . This contrasts with known examples where such inequality does not hold under weaker conditions on the ambient group G or the subgroup H . We include applications to hyperbolic groups and homotopical filling functions.

1. INTRODUCTION

The n^{th} homological and homotopical filling functions of a space are generalized isoperimetric functions describing the minimal volume required to fill an n -cycle or n -sphere with an $(n+1)$ -chain or $(n+1)$ -ball. These functions have been widely studied in Riemannian Geometry and Geometric Group Theory; see for example [2, 5, 8, 10, 15, 18]. In this paper, we study the relation between the n^{th} homological filling functions of a finitely presented group and its subgroups. Our main result provides sufficient conditions for the n^{th} -filling function of a subgroup to be bounded from above by the n^{th} -filling function of the ambient group. The hypotheses of our theorem are in terms of finiteness properties of the ambient group and the subgroup. Our result contrasts with known examples illustrating that this relation does not hold under weaker conditions [4, 22, 21].

1.1. Statement of Main Result. A $K(G, 1)$ for a group G is a cell complex X with contractible universal cover \tilde{X} and fundamental group isomorphic to G . If G admits a $K(G, 1)$ with finite n -skeleton, then G is said to be of type F_n . Such finiteness properties are natural (topological) generalizations of being finitely generated (type F_1) and finitely presented (type F_2).

If X is a $K(G, 1)$ with finite $(n+1)$ -skeleton, then the n^{th} -homological filling function of G is an optimal function $FV_G^{n+1} : \mathbb{N} \rightarrow \mathbb{N}$ such that $FV_G^{n+1}(k)$ bounds the minimal volume required to fill an n -cycle γ of \tilde{X} of volume at most k , with an $(n+1)$ -chain μ of \tilde{X} having boundary $\partial(\mu) = \gamma$. See Section 3 for precise definitions.

It can be shown that the growth rate of FV_G^{n+1} is independent of the choice of X up to an equivalence relation \sim , hence FV_G^{n+1} is an invariant of the group G , see [9, 20]. The relation $f \sim g$ between functions is defined as $f \leq g$ and $g \leq f$, where $f \leq g$ means that there is $C > 0$ such that for all $n \in \mathbb{N}$, $f(n) \leq Cg(Cn + C) + Cn + C$. Our main result is a generalization of a result of Gersten [12, Thm C] to higher dimensions.

Theorem 1.1. *Let $n \geq 1$. Let G be a group admitting a finite $(n+1)$ -dimensional $K(G, 1)$ and let $H \leq G$ be a subgroup of type F_{n+1} . Then*

$$FV_H^{n+1} \leq FV_G^{n+1}.$$

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Some examples that contrast with Theorem 1.1 are the following. In [4], Noel Brady constructed a group G admitting a finite 3-dimensional $K(G, 1)$ such that FV_G^2 is linear, and G contains a subgroup $H \leq G$ of type F_2 with FV_H^2 at least quadratic. Another source of examples are the generalized Heisenberg groups \mathcal{H}_{2n+1} , for which Robert Young computed the homological filling invariants in [22, 21]. For instance, \mathcal{H}_5 admits a finite 5-dimensional $K(\mathcal{H}_5, 1)$ and has quadratic $FV_{\mathcal{H}_5}^2$. On the other hand, \mathcal{H}_3 can be embedded in \mathcal{H}_5 , admits a 3-dimensional $K(\mathcal{H}_3, 1)$, and has cubic $FV_{\mathcal{H}_3}^2$. Likewise, \mathcal{H}_5 has quadratic $FV_{\mathcal{H}_5}^3$ and can be embedded in \mathcal{H}_7 which has $FV_{\mathcal{H}_7}^3$ polynomial of degree $3/2$.

Theorem 1.1 also imposes constraints on certain well known constructions. For example, given a finitely generated group H with decidable word problem in nondeterministic polynomial time, Birget, Ol’shanskii, Rips and Sapir produce an embedding of H into a finitely presented group G with polynomial Dehn function [3]. For this construction, Theorem 1.1 implies that if H has a finite 2-dimensional $K(H, 1)$ and FV_H^2 is not bounded by a polynomial function, then G does not admit a finite 2-dimensional $K(G, 1)$. A particular example of such a group H is the Baumslag-Solitar group $B(m, n)$ with $|m| \neq |n|$, for which the embedding constraint is known [12, Thm A].

We discuss some applications of Theorem 1.1 to hyperbolic groups and homotopical filling functions below. Recall that a group G is *hyperbolic* if it has a linear Dehn function. In [13], Gersten proved the following:

Theorem 1.2. [13, Thm 4.6] *Let G be a hyperbolic group of cohomological dimension 2. Then every finitely presented subgroup $H \leq G$ is hyperbolic.*

Gersten’s result does not hold in higher dimensions as Brady has exhibited a hyperbolic group G of cohomological dimension 3 containing a non-hyperbolic finitely presented subgroup $H \leq G$ [4]. We can however, obtain a result similar to Theorem 1.2 by considering homotopical filling functions of higher dimensions. The n^{th} -homotopical filling function δ_G^n of a group G is defined analogously to FV_G^{n+1} but restricts to filling n -spheres with $(n+1)$ -balls inside the universal cover of $K(G, 1)$ with finite $(n+1)$ -skeleton. Roughly speaking, $\delta_G^n(k)$ bounds the minimum volume required to fill an n -sphere of volume at most k , with an $(n+1)$ -ball. Precise definitions of “volume” and δ_G^n can be found in [2, 5].

Corollary 1.3. *Let G be a hyperbolic group of geometric dimension $n+1$, where $n \geq 2$. Let $H \leq G$ be of type F_{n+1} . Then δ_H^n is linear.*

Recall that the *geometric dimension* of a group G is the minimum dimension among $K(G, 1)$ ’s. The Eilenberg–Ganea Theorem [6, 7] states that the cohomological and geometric dimensions of a group G are equal for dimensions greater or equal than 3. This justifies our use of geometric dimension in the corollary above. In addition to Corollary 1.3, we have the following homotopical version of Theorem 1.1 for sufficiently large n .

Corollary 1.4. *Let $n \geq 3$. Let G be a group admitting a finite $(n+1)$ -dimensional $K(G, 1)$. Let $H \leq G$ be of type F_{n+1} . Then $\delta_H^n \leq \delta_G^n$.*

Corollaries 1.3 and 1.4 follow from Theorem 1.1 and the following results:

Theorem 1.5. [1, pg. 1 and references therein] *For $n \geq 3$, the n^{th} -homotopical and homological filling functions δ_G^n and FV_G^{n+1} are equivalent. For $n = 2$, $\delta_G^2 \leq FV_G^3$.*

Theorem 1.6. [17] *Let G be a hyperbolic group. Then FV_G^{n+1} is linear for all $n \geq 1$.*

Proof of Corollary 1.3. A theorem of Rips implies that G admits a compact $K(G, 1)$, see [14] and then the Eilenberg–Ganea Theorem implies that G admits a compact $(n+1)$ -dimensional

$K(G, 1)$, see [6]. Theorems 1.1 and 1.6 imply that FV_H^{n+1} is linear. It then follows from Theorem 1.5 that δ_H^n is also linear. \square

Proof of Corollary 1.4. Apply Theorems 1.5 and 1.1. \square

Remark 1.7. *Corollary 1.3 does not apply to Brady's example $H \leq G$ mentioned above since H is not of type F_3 . It is an open question whether or not the subgroups H in Corollary 1.3 are in fact hyperbolic.*

Remark 1.8. *It is an open question whether or not the statement of Corollary 1.4 holds for $n = 1$ or 2 . In general $\delta_G^1 \neq FV_G^2$ and $\delta_G^2 \neq FV_G^3$, examples of such groups are given in [1, 20].*

1.2. Outline of the Paper. The rest of the paper is organized into three sections. Section 2 contains the definition of a filling norm on a finitely generated $\mathbb{Z}G$ -module and lemmas required for the proof of Theorem 1.1. Section 3 contains algebraic and topological definitions for FV_G^{n+1} . Section 4 contains the proof of Theorem 1.1.

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2. FILLING NORMS ON $\mathbb{Z}G$ -MODULES

In this section we define the notion of a filling-norm on a finitely generated $\mathbb{Z}G$ -module. Most ideas in this section are based on the work of Gersten in [13]. The section contains four lemmas on which the proof of the main result of the paper relies on.

Definition 2.1 (Norm on Abelian Groups). *A norm on an Abelian group A is a function $\|\cdot\|: A \rightarrow \mathbb{R}$ satisfying the following conditions:*

- $\|a\| \geq 0$ with equality if and only if $a = 0$, and
- $\|a\| + \|a'\| \geq \|a + a'\|$.

If, in addition, the norm satisfies

- $\|na\| = |n| \cdot \|a\|$, for $n \in \mathbb{Z}$,

then it is called a regular norm.

If A is free Abelian with basis X , then X induces a regular ℓ_1 -norm on A given by

$$\left\| \sum_{x \in X} n_x x \right\|_1 = \sum_{x \in X} |n_x|, \text{ where } n_x \in \mathbb{Z}.$$

Definition 2.2 (Linearly Equivalent Norms). *Two norms $\|\cdot\|$ and $\|\cdot\|'$ on a \mathbb{Z} -module M are linearly equivalent if there exists a fixed constant $C > 0$ such that*

$$C^{-1}\|m\| \leq \|m\|' \leq C\|m\|$$

for all $m \in M$. This is an equivalence relation and the equivalence class of a norm $\|\cdot\|$ is called the linear equivalence class of $\|\cdot\|$.

Definition 2.3 (Based Free $\mathbb{Z}G$ -modules and Induced ℓ_1 -norms). *Suppose G is a group and F is a free $\mathbb{Z}G$ -module with $\mathbb{Z}G$ -basis $\{\alpha_1, \dots, \alpha_n\}$. Then $\{g\alpha_i : g \in G, 1 \leq i \leq n\}$ is a free \mathbb{Z} -basis for F as a (free) \mathbb{Z} -module. This free \mathbb{Z} -basis induces a G -equivariant ℓ_1 -norm $\|\cdot\|_1$ on F . We call a free $\mathbb{Z}G$ -module based if it is understood to have a fixed basis, and we use this basis for the induced ℓ_1 -norm $\|\cdot\|_1$.*

Definition 2.4 (Filling Norms on $\mathbb{Z}G$ -modules). *Let $\eta: F \rightarrow M$ be a surjective homomorphism of $\mathbb{Z}G$ -modules and suppose that F is free, finitely generated, and based. The filling norm on M induced by η and the free $\mathbb{Z}G$ -basis of F is defined as*

$$\|m\|_\eta = \min \{ \|x\|_1 : x \in F, \eta(x) = m \}.$$

Observe that this norm is G -equivariant.

Remark 2.5. *Gersten observed that filling norms are not in general regular norms. He illustrated this fact with the following example [11]. Let X be the universal cover of the standard complex of the group presentation $\langle x|x^2, x^{2k} \rangle$, where $k \geq 2$. The filling norm on the integral cellular 1-cycles $Z_1(X)$ induced by $C_2(X) \xrightarrow{\partial} Z_1(X)$ is not regular since $\|2x\|_\partial = \|2kx\|_\partial = 1$.*

Remark 2.6 (Induced ℓ_1 -norms are Filling Norms). *If F is a finitely generated based free $\mathbb{Z}G$ -module, then the ℓ_1 -norm induced by a free $\mathbb{Z}G$ -basis is a filling norm.*

The following lemma is reminiscent of the fact that linear operators on finite dimensional normed spaces are bounded.

Lemma 2.7 ($\mathbb{Z}G$ -Morphisms between Free Modules are Bounded). [13, Proof of Proposition 4.4] *Let $\varphi: F \rightarrow F'$ be a homomorphism between finitely generated, free, based $\mathbb{Z}G$ -modules. Let $\|\cdot\|_1$ and $\|\cdot\|'_1$ denote the induced ℓ_1 -norms of F and F' . Then there exists a constant $C > 0$ such that for all $x \in F$*

$$\|\varphi(x)\|'_1 \leq C \cdot \|x\|_1.$$

Proof. Let $A = \{\alpha_1, \dots, \alpha_n\}$ be the $\mathbb{Z}G$ -basis of F inducing the norm $\|\cdot\|_1$. Then φ is given by a finite $n \times m$ matrix whose entries are elements of $\mathbb{Z}G$. Observe that for any $g \in G$, $x \in F$, we have $\|x\|_1 = \|gx\|_1$. Define $C = \max_{1 \leq i \leq n} \{\|\varphi(\alpha_i)\|'_1\}$ and let $x \in F$ be arbitrary.

Then

$$\begin{aligned} \|\varphi(x)\|'_1 &= \|\varphi(\lambda_1 \alpha_1 + \dots + \lambda_n \alpha_n)\|'_1, \text{ where } \lambda_i \in \mathbb{Z}G \\ &\leq \left\| \left(\sum_{g \in G} \lambda_{1,g} g \right) \varphi(\alpha_1) \right\|'_1 + \dots + \left\| \left(\sum_{g \in G} \lambda_{n,g} g \right) \varphi(\alpha_n) \right\|'_1, \text{ where } \lambda_j = \sum_{g \in G} \lambda_{j,g} g \text{ and } \lambda_{j,g} \in \mathbb{Z} \\ &\leq \left(\sum_{g \in G} |\lambda_{1,g}| \right) \|\varphi(\alpha_1)\|'_1 + \dots + \left(\sum_{g \in G} |\lambda_{n,g}| \right) \|\varphi(\alpha_n)\|'_1 \\ &\leq C \left(\sum_{i=1}^n \left(\sum_{g \in G} |\lambda_{i,g}| \right) \right) = C \|x\|_1. \end{aligned} \quad \square$$

Lemma 2.8 ($\mathbb{Z}G$ -Morphisms with Projective Domain are Bounded). *Let $\varphi: P \rightarrow Q$ be a homomorphism between finitely generated $\mathbb{Z}G$ -modules. Let $\|\cdot\|_P$ and $\|\cdot\|_Q$ denote filling norms on P and Q respectively. If P is projective then there exists a constant $C > 0$ such that for all $p \in P$*

$$\|\varphi(p)\|_Q \leq C \cdot \|p\|_P.$$

Proof. Consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\tilde{\varphi}} & B \\ \rho \downarrow & \nearrow \psi & \downarrow \\ P & \xrightarrow{\varphi} & Q \end{array}$$

constructed as follows. Let A and B be finitely generated and based free $\mathbb{Z}G$ -modules, and let $A \rightarrow P$ and $B \rightarrow Q$ be surjective morphisms inducing the filling norms $\|\cdot\|_P$ and $\|\cdot\|_Q$. Since P is projective and $B \rightarrow Q$ is surjective, there is a lifting $\psi: P \rightarrow B$ of φ ; then let $\tilde{\varphi}$ be the composition $A \xrightarrow{\rho} P \xrightarrow{\psi} B$. Let C be the constant provided by Lemma 2.7 for $\tilde{\varphi}$. Let $p \in P$ and let $a \in A$ that maps to p . It follows that

$$\|\varphi(p)\|_Q \leq \|\psi(p)\|_1 = \|\tilde{\varphi}(a)\|_1 \leq C\|a\|_1.$$

Since the above inequality holds for any $a \in A$ with $\rho(a) = p$, it follows that

$$\begin{aligned} \|\varphi(p)\|_Q &\leq C \cdot \min_{\rho(a)=p} \{\|a\|_1\} \\ &= C \cdot \|p\|_P. \end{aligned} \quad \square$$

Lemma 2.9 (Equivalence of Filling Norms for $\mathbb{Z}G$ -modules). [13, Lemma 4.1] *Any two filling norms on a finitely generated $\mathbb{Z}G$ -module M are linearly equivalent.*

Proof. Consider a pair of surjective homomorphisms of $\mathbb{Z}G$ -modules $\eta: F \rightarrow M$ and $\eta': F' \rightarrow M$ such that F and F' are finitely generated, free, based modules inducing the filling norms $\|\cdot\|_\eta$ and $\|\cdot\|_{\eta'}$ on M . Since η' is surjective, the universal property of F provides a homomorphism φ such that $\eta = \eta' \circ \varphi$. Let $m \in M$ be arbitrary and take $x \in F$ such that $\eta(x) = m$. Since $\eta' \circ \varphi(x) = m$, by Lemma 2.7 there exists $C > 0$ such that

$$\|m\|_{\eta'} = \min_{\eta'(x')=m} \|x'\|_1 \leq \|\varphi(x)\|_1' \leq C \cdot \|x\|_1.$$

As this inequality holds for all $x \in F$ satisfying $\eta(x) = m$, we have

$$\|m\|_{\eta'} \leq C \cdot \min_{\eta(x)=m} \{\|x\|_1\} = C \cdot \|m\|_\eta.$$

The other inequality proceeds in a similar manner. \square

Lemma 2.10 (Retraction Lemma). [13, Prop. 4.4] *Let $0 \rightarrow M \xrightarrow{\iota} N \rightarrow P \rightarrow 0$ be a short exact sequence of $\mathbb{Z}G$ -modules where*

- 1) M is finitely generated and equipped with a filling-norm $\|\cdot\|_M$.
- 2) N is free, based, and equipped with the induced ℓ_1 -norm $\|\cdot\|_1$.
- 3) P is projective.

Then there exists a retraction $\rho: N \rightarrow M$ for the inclusion $\iota: M \rightarrow N$ and a fixed constant $C > 0$ such that $\|\rho(x)\|_M \leq C\|x\|_1$ for all $x \in N$.

Proof. Since P is projective there is a retraction ρ' for ι . Since M is finitely generated, N is isomorphic to a product $I \oplus Q$ of free modules where I is finitely generated and contains the image of M . Define $\rho: N \rightarrow M$ by $\rho|_I = \rho'|_I$ and $\rho|_Q = 0$. Then ρ is a retraction for ι with support contained in I .

Each $x \in N$ has a unique decomposition $x = y + q$ where $y \in I$, $q \in Q$ such that $\rho(x) = \rho(y)$ and $\|y\|_1 \leq \|x\|_1$. Apply Lemma 2.8 to the restriction $\rho: I \rightarrow M$ to obtain $C > 0$ such that

$$\|\rho(x)\|_M = \|\rho(y)\|_M \leq C\|y\|_1 \leq C\|x\|_1.$$

\square

3. HOMOLOGICAL FILLING FUNCTIONS OF GROUPS

In this section, given a group G of type FP_{n+1} , where $n \geq 1$, we define the group invariant FV_G^{n+1} . In the first part of the section we provide an algebraic definition of FV_G^{n+1} and prove that it is well defined. This algebraic approach, while naturally inspired by the topological approach, provides a convenient algebraic framework suitable for some of the arguments in this paper. This algebraic approach has been also explored in [16]. In the second part, we recall the topological approach to FV_G^{n+1} and show that the topological and algebraic approaches are equivalent for finitely presented groups of type FP_{n+1} . The final subsection discusses why $FV_G^{n+1}(k)$ is a finite number.

3.1. Algebraic Definition of FV_G^{n+1} .

Definition 3.1 (Linearly Equivalent Functions). *Let f and g be functions from \mathbb{N} to \mathbb{N} . Define $f \leq g$ if there exists $C > 0$ such that for all $n \in \mathbb{N}$*

$$f(n) \leq Cg(Cn + C) + Cn + C.$$

The functions f and g are linearly equivalent, $f \sim g$, if both $f \leq g$ and $g \leq f$ hold. This is an equivalence relation and the equivalence class containing a function f is called the linear equivalence class of f .

Definition 3.2 (FP_n group). [6] *A group G is of type FP_n if there is a resolution of $\mathbb{Z}G$ -modules*

$$P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \longrightarrow \mathbb{Z} \rightarrow 0,$$

such that for each $i \in \{0, 1, \dots, n\}$ the module P_i is a finitely generated projective $\mathbb{Z}G$ -module. In this case, such a resolution is called an FP_n -resolution.

Definition 3.3 (Algebraic definition of FV_G^{n+1}). *Let G be a group of type FP_{n+1} . The algebraic n^{th} -filling function is the (linear equivalence class of the) function*

$$FV_G^{n+1} : \mathbb{N} \rightarrow \mathbb{N}$$

defined as follows. Let

$$P_{n+1} \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} \dots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \longrightarrow \mathbb{Z} \rightarrow 0,$$

be a resolution of $\mathbb{Z}G$ -modules for \mathbb{Z} of type FP_{n+1} . Choose filling norms for P_n and P_{n+1} , denoted by $\|\cdot\|_{P_n}$ and $\|\cdot\|_{P_{n+1}}$ respectively. Then

$$FV_G^{n+1}(k) = \max \{ \|\gamma\|_{\partial_{n+1}} : \gamma \in \ker(\partial_n), \|\gamma\|_{P_n} \leq k \},$$

where

$$\|\gamma\|_{\partial_{n+1}} = \min \{ \|\mu\|_{P_{n+1}} : \mu \in P_{n+1}, \partial_{n+1}(\mu) = \gamma \}.$$

Remark 3.4 (Finiteness of FV_G^{n+1}). *It is not immediately clear that the maximum in Definition 3.3 is a finite number. In Section 3.3 we recall some results from the literature which, under the assumption G is finitely presented, imply that FV_G^{n+1} is a finite valued function for $n = 1$ and $n \geq 3$. The authors are not aware of a proof for the case $n = 2$.*

For $n = 2$, all results in this paper regarding FV_G^3 hold under the following natural modifications. First, work with the standard extensions of addition, multiplication, and order, of the positive integers \mathbb{N} to $\mathbb{N} \cup \{\infty\}$. Definition 3.1 is extended to functions $\mathbb{N} \rightarrow \mathbb{N} \cup \{\infty\}$, but we emphasize that the constant C remains a finite positive integer. In Definition 3.3 the function FV_G^3 is defined as an $\mathbb{N} \rightarrow \mathbb{N} \cup \{\infty\}$ function. We observe that no argument in this paper relies on $FV_G^{n+1}(k)$ being finite.

Theorem 3.5 (FV_G^{n+1} is a Well-defined Group Invariant). *Let G be a group of type FP_{n+1} . Then the algebraic n^{th} -filling function FV_G^{n+1} of G is well defined up to linear equivalence.*

Proof. Let (F_*, ∂_*) and (P_*, δ_*) be a pair of resolutions of $\mathbb{Z}G$ -modules of type FP_{n+1} with choices of filling-norms for their n^{th} and $(n+1)^{\text{th}}$ modules denoted by $\|\cdot\|_{F_n}$ and $\|\cdot\|_{F_{n+1}}$, and $\|\cdot\|_{P_n}$ and $\|\cdot\|_{P_{n+1}}$ respectively. Let $FV_{F_*}^{n+1}$ and $FV_{P_*}^{n+1}$ be the induced functions according to Definition 3.3. By symmetry, it is enough to show that $FV_{F_*}^{n+1} \leq FV_{P_*}^{n+1}$.

It is well known that any two projective resolutions of a $\mathbb{Z}G$ -module are chain homotopy equivalent, see for example [6, pg.24, Thm 7.5], and hence the resolutions F_* and P_* are chain homotopy equivalent. Therefore there exists chain maps $f_i : F_i \rightarrow P_i$, $g_i : P_i \rightarrow F_i$, and a map $h_i : F_i \rightarrow F_{i+1}$ such that

$$\partial_{i+1} \circ h_i + h_{i-1} \circ \partial_i = g_i \circ f_i - id.$$

Let C denote the maximum of the constants for the maps g_{n+1} , h_n , and f_n and the chosen filling-norms provided by Lemma 2.8. We claim that for every $k \in \mathbb{N}$,

$$FV_{F_*}^{n+1}(k) \leq C \cdot FV_{P_*}^{n+1}(Ck + C) + Ck + C.$$

Fix k . Let $\alpha \in \ker(\partial_n)$ be such that $\|\alpha\|_{F_n} \leq k$. Choose $\beta \in P_{n+1}$ such that $\delta_{n+1}(\beta) = f_n(\alpha)$ and $\|f_n(\alpha)\|_{\delta_{n+1}} = \|\beta\|_{P_{n+1}}$. By commutativity of the chain maps and the chain homotopy equivalence,

$$\begin{aligned} \partial_{n+1} \circ h_n(\alpha) + h_{n-1} \circ \partial_n(\alpha) &= g_n \circ f_n(\alpha) - \alpha \\ &= g_n \circ \delta_{n+1}(\beta) - \alpha \\ &= \partial_{n+1} \circ g_{n+1}(\beta) - \alpha. \end{aligned}$$

Since $\alpha \in \ker(\partial_n)$, we have that $h_{n-1} \circ \partial_n(\alpha) = 0$. Rearranging the above equation, we obtain

$$\alpha = \partial_{n+1} \circ g_{n+1}(\beta) - \partial_{n+1} \circ h_n(\alpha) = \partial_{n+1}(g_{n+1}(\beta) - h_n(\alpha)).$$

Hence $g_{n+1}(\beta) - h_n(\alpha)$ has boundary α . Observe that

$$\begin{aligned} \|\alpha\|_{\partial_{n+1}} &\leq \|g_{n+1}(\beta) - h_n(\alpha)\|_{F_{n+1}} && \text{since } \partial_{n+1}(g_{n+1}(\beta) - h_n(\alpha)) = \alpha \\ &\leq \|g_{n+1}(\beta)\|_{F_{n+1}} + \|h_n(\alpha)\|_{F_{n+1}} && \text{by the triangle inequality} \\ &\leq C \cdot \|\beta\|_{P_{n+1}} + C \cdot \|\alpha\|_{F_n} && \text{by Lemma 2.8} \\ &= C \cdot \|f_n(\alpha)\|_{\delta_{n+1}} + C \cdot \|\alpha\|_{F_n} && \text{by definition of } \beta \\ &\leq C \cdot FV_{P_*}^{n+1}(\|f_n(\alpha)\|_{P_n}) + C\|\alpha\|_{F_n} && \text{by definition of } FV_{P_*}^{n+1} \\ &\leq C \cdot FV_{P_*}^{n+1}(C\|\alpha\|_{F_n}) + C\|\alpha\|_{F_n} && \text{by Lemma 2.8} \\ &\leq C \cdot FV_{P_*}^{n+1}(Ck + C) + Ck + C && \text{since } \|\alpha\|_{F_n} \leq k. \end{aligned}$$

Since α was arbitrary, $FV_{F_*}^{n+1}(k) \leq C \cdot FV_{P_*}^{n+1}(Ck + C) + Ck + C$ for all $k \in \mathbb{N}$. This shows that $FV_{F_*}^{n+1} \leq FV_{P_*}^{n+1}$ completing the proof. \square

3.2. Topological Definition of FV_G^{n+1} . For a cell complex X , the cellular chain group $C_i(X)$ is a free Abelian group with basis the collection of all i -cells of X . This natural basis induces an ℓ_1 -norm on $C_i(X)$ that we denote by $\|\cdot\|_1$. Recall that a complex X is n -connected if its first n -homotopy groups are trivial.

Definition 3.6 (F_n group). *A group G is of type F_n if there is a $K(G, 1)$ -complex with a finite n -skeleton, i.e., with only finitely many cells in dimensions $\leq n$.*

Definition 3.7 (Topological Definition of FV_G^{n+1}). [9, 20] *Let G be a group acting properly, cocompactly, by cellular automorphisms on an n -connected cell complex X . The topological n^{th} -filling function of G is the (linear equivalence class of the) function $FV_G^{n+1} : \mathbb{N} \rightarrow \mathbb{N}$ defined as*

$$FV_G^{n+1}(k) = \max \{ \|\gamma\|_{\partial} : \gamma \in Z_n(X), \|\gamma\|_1 \leq k \},$$

where

$$\|\gamma\|_{\partial} = \min \{ \|\mu\|_1 : \mu \in C_{n+1}(X), \partial(\mu) = \gamma \}.$$

J. Fletcher and R. Young have independently provided geometric proofs that the topological n^{th} -filling function FV_G^{n+1} is well defined as an invariant of the group, see [9, Theorem 2.1] and [20, Lemma 1] respectively. In the work of Fletcher, the topological definition of FV_G^{n+1} requires X to be the universal cover of a $K(G, 1)$, while Young's proof is in the more general context introduced above.

Theorem 3.8. [20] *Let G be a group admitting a proper and cocompact action by cellular automorphisms on an n -connected cell complex. Then the topological n^{th} -filling invariant FV_G^{n+1} of G is well defined up to linear equivalence.*

Even in the topological definition, it is not trivial that FV_G^{n+1} is a finite valued function and Remark 3.4 also applies in this case. For the rest of the section, we show that the topological and algebraic approaches to FV_G^{n+1} are equivalent for finitely presented groups of type FP_{n+1} .

Proposition 3.9. *Let $n \geq 1$ and let G be a group of type F_{n+1} . Then G is of type FP_{n+1} and the algebraic and topological n^{th} -filling functions of G are linearly equivalent.*

Proof. Let X be a $K(G, 1)$ with finite $(n+1)$ -skeleton. The G -action on the universal cover \tilde{X} of X induces the structure of a $\mathbb{Z}G$ -module to the group of cellular chains $C_i(\tilde{X})$ and each boundary map ∂_i is a morphism of $\mathbb{Z}G$ -modules. Since the G -action on \tilde{X} is cellular and free, each $C_i(\tilde{X})$ is a free $\mathbb{Z}G$ -module with basis any collection of representatives of the G -orbits of i -cells. Since the action is cocompact on the $(n+1)$ -skeleton, each $C_i(\tilde{X})$ is a finitely generated free $\mathbb{Z}G$ -module for $i \in \{0, 1, \dots, n+1\}$. Since \tilde{X} is a contractible space, all its homology groups are trivial and therefore we have a resolution of $\mathbb{Z}G$ -modules

$$\cdots \longrightarrow C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \longrightarrow \mathbb{Z} \rightarrow 0,$$

of type FP_{n+1} . Under our assumptions, the induced topological n^{th} -filling function of G is a particular instance of an algebraic n^{th} -filling function of G . The conclusion then follows from Theorems 3.5 and 3.8. \square

Proposition 3.10. [6, pg 205, proof of Thm. 7.1] *Let G be finitely presented and of type FP_n where $n \geq 2$. Then G is of type F_n .*

Propositions 3.9 and 3.10 imply the following statement.

Corollary 3.11. *Let G be a finitely presented group of type FP_{n+1} . Then the topological and algebraic definitions of FV_G^{n+1} are equivalent.*

3.3. Finiteness of $FV_G^{n+1}(k)$. Let G be a finitely presented group of type FP_{n+1} , or equivalently assume that G is of type F_{n+1} ; see Proposition 3.10. We will sketch why FV_G^{n+1} is a finite valued function for $n = 1$ and $n \geq 3$.

3.3.1. *Case $n = 1$.* Finiteness of FV_G^2 follows from that of the Dehn function δ_G . We summarize the argument from Gersten's article [10, Prop 2.4]. Let X be a $K(G, 1)$ with finite 2-skeleton and let $z \in Z_1(\widetilde{X})$ be a 1-cycle with $\|\gamma\|_1 \leq k$. Then $z = z_1 + \dots + z_m$ for some $m \leq k$ where each z_i is the 1-cycle induced by a simple edge circuit γ_i in \widetilde{X} and $\sum_{i=1}^m \ell(\gamma_i) = \|z\|_1$. Then

$$\|z\|_{\partial_2} \leq \sum_{i=1}^m \text{Area}(\gamma_i) \leq \sum_{i=1}^m \delta_G(\ell(\gamma_i)) \leq k \cdot \delta_G(k) < \infty.$$

3.3.2. *Case $n \geq 3$.* A group G of type F_{n+1} has a well defined invariant called the n^{th} -homotopical filling function $\delta_G^n: \mathbb{N} \rightarrow \mathbb{N}$. There are multiple approaches to define δ_G^n , we sketch the approach found in [1, 5]. Roughly speaking, if X is a $K(G, 1)$ with finite $(n+1)$ -skeleton, then $\delta_G^n(k)$ measures the number of $(n+1)$ -balls required to fill a sphere $S^n \rightarrow \widetilde{X}$ comprised of at most k n -balls. Here the maps $f: S^n \rightarrow \widetilde{X}$ and fillings $\tilde{f}: D^{n+1} \rightarrow \widetilde{X}$ are required to be in a particular class of maps called *admissible maps*. This allows one to define the volumes, $\text{vol}(f)$ and $\text{vol}(\tilde{f})$, as the number of n -balls and $(n+1)$ -balls of S^n and D^{n+1} respectively, mapping homeomorphically to open cells of \widetilde{X} . The *filling volume* of f is given by

$$\text{FVol}(f) = \sup\{ \text{vol}(\tilde{f}) \mid \tilde{f}: D^{n+1} \rightarrow \widetilde{X}, \tilde{f}|_{\partial D^{n+1}} = f \}$$

and δ_G^n by

$$\delta_G^n(k) = \max\{ \text{FVol}(f) \mid f: S^n \rightarrow \widetilde{X}, \text{vol}(f) \leq k \}.$$

Alonso et al. use higher homotopy groups as $\pi_1(X)$ -modules to provide a more algebraic approach to δ_G^n , in particular they show that δ_G^n is a finite valued function [2, Corollary 1]. It is observed in [5, Remark 2.4(2)] that Alonso's approach and the approach described above are equivalent.

The finiteness of FV_G^{n+1} then follows from the inequality

$$FV_G^{n+1} \leq \delta_G^n$$

which holds for all $n \geq 3$. We outline the argument for this inequality described in the introduction of [1]. Let X be a $K(G, 1)$ with finite $(n+1)$ -skeleton and let $\gamma \in Z_n(\widetilde{X})$ with $\|\gamma\|_1 \leq k$. Using the Hurewicz Theorem, one can show (see [15, 19]) that γ is the image of the fundamental class of an n -sphere for a map $f: S_n \rightarrow \widetilde{X}$ such that $\text{vol}(f) = \|\gamma\|_1$. If $\tilde{f}: D^{n+1} \rightarrow \widetilde{X}$ is an extension of f to the $(n+1)$ -ball D^{n+1} , then the image of the fundamental class of D^{n+1} is an $(n+1)$ -chain μ with $\partial(\mu) = \gamma$ and $\text{vol}(\tilde{f}) \geq \|\mu\|_1$. Therefore the filling volume

$$\text{FVol}(f) = \sup\{ \text{vol}(\tilde{f}) \mid \tilde{f}: D^{n+1} \rightarrow \widetilde{X}, \tilde{f}|_{\partial D^{n+1}} = f \}$$

is greater than or equal to the filling norm $\|\gamma\|_{\partial_{n+1}}$. It follows that $FV_G^{n+1}(k) \leq \delta_G^n(k)$

4. MAIN RESULT

As we will be working with cell complexes, all relevant computations in this section are understood to occur within cellular chain complexes.

Definition 4.1 (Stably free). *A $\mathbb{Z}G$ -module P is stably free if there exists finitely generated free $\mathbb{Z}G$ module F such that $P \oplus F$ is free.*

Lemma 4.2 (The Eilenberg Trick). [6, pg.207] *Let $G = \pi_1(X, x_0)$, where X is a cell complex. Then X is a subcomplex of a complex Y such that the inclusion $X \hookrightarrow Y$ is a homotopy equivalence, and the cellular n -cycles of the universal covers \tilde{Y} and \tilde{X} satisfy $Z_n(\tilde{Y}) \simeq Z_n(\tilde{X}) \oplus \mathbb{Z}G$ as $\mathbb{Z}G$ -modules.*

Proof. Let x_0 be a 0-cell of X , and glue an n -cell D^n to (X, x_0) by mapping its boundary to x_0 . The resulting space is the wedge sum of X and an n -sphere S^n . To obtain Y , attach an $(n+1)$ -cell D^{n+1} by the attaching map that identifies ∂D^{n+1} with the n -sphere S^n . Then $Z_n(\tilde{Y}) \simeq Z_n(\tilde{X}) \oplus \mathbb{Z}G$ where the $\mathbb{Z}G$ factor is generated by a lifting of the n -cell D^n to \tilde{Y} . It is clear that $X \hookrightarrow Y$ is a homotopy equivalence. \square

Lemma 4.3 (Schanuel's Lemma). [6, pg.193, Lemma 4.4] *Let*

$$0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0$$

and

$$0 \rightarrow P'_n \rightarrow P'_{n-1} \rightarrow \cdots \rightarrow P'_0 \rightarrow M \rightarrow 0$$

be exact sequences of R -modules with P_i and P'_i projective for $i \leq n-1$. Then

$$P_0 \oplus P'_1 \oplus P_2 \oplus P'_3 \oplus \cdots \simeq P'_0 \oplus P_1 \oplus P'_2 \oplus P_3 \oplus \cdots$$

We are now ready to prove our main result which is a generalization of [12, Thm C]. The proof is based on Gersten's proof of [13, Thm 4.6] and is adjusted for higher dimensions:

Theorem 4.4. *Let G be a group admitting a finite $(n+1)$ -dimensional $K(G, 1)$ and let $H \leq G$ be a subgroup of type F_{n+1} . Then $FV_H^{n+1} \leq FV_G^{n+1}$.*

Proof. Let W be a finite $(n+1)$ -dimensional $K(G, 1)$. Let X be the $(n+1)$ -skeleton of a $K(H, 1)$. Since H is of type F_{n+1} , we may assume that X is a finite cell complex. Then, after subdivisions, there exists a cellular map $f : X \rightarrow W$ inducing the inclusion $H \hookrightarrow G$ at the level of fundamental groups. Let M_f be the mapping cylinder of f and consider the exact sequences of $\mathbb{Z}G$ -modules

$$(4.1) \quad 0 \rightarrow Z_n(\tilde{M}_f) \rightarrow C_n(\tilde{M}_f) \rightarrow \cdots \rightarrow C_0(\tilde{M}_f) \rightarrow \mathbb{Z} \rightarrow 0$$

and

$$(4.2) \quad 0 \rightarrow C_{n+1}(\tilde{W}) \rightarrow C_n(\tilde{W}) \rightarrow \cdots \rightarrow C_0(\tilde{W}) \rightarrow \mathbb{Z} \rightarrow 0,$$

where \tilde{W} and \tilde{M}_f denote the universal covers of W and M_f respectively.

Applying Schanuel's lemma to the above sequences shows that the $\mathbb{Z}G$ -module $Z_n(\tilde{M}_f)$ is finitely generated and stably free. Let Y be the space obtained by attaching a finite number of $(n+1)$ -balls to the base point of M_f as in Lemma 4.2 such that $Z_n(\tilde{Y})$ is finitely generated and free as a $\mathbb{Z}G$ -module.

From here on, we are only concerned with the inclusion map $X \rightarrow Y$ realizing the inclusion $H \rightarrow G$ at the level of fundamental groups with the property that $Z_n(\tilde{Y})$ is finitely generated and free as a $\mathbb{Z}G$ -module. Since the inclusion $X \rightarrow Y$ is injective at the level of fundamental groups, any lifting $\tilde{X} \rightarrow \tilde{Y}$ is an embedding. Moreover, we can choose the lifting to be equivariant with respect to the inclusion $H \rightarrow G$. Without loss of generality, assume that \tilde{X} is an H -equivariant subcomplex of \tilde{Y} .

Since the ring $\mathbb{Z}G$ is free as a $\mathbb{Z}H$ -module, it follows that $C_i(\tilde{Y})$ is a free $\mathbb{Z}H$ -module. Since \tilde{X} is an H -equivariant subcomplex of \tilde{Y} , the $\mathbb{Z}H$ -module $C_i(\tilde{X})$ is a free factor of $C_i(\tilde{Y})$. Hence the quotient $C_i(\tilde{Y}^{(n)}, \tilde{X}^{(n)}) = C_i(\tilde{Y})/C_i(\tilde{X})$ is a free $\mathbb{Z}H$ -module.

Restricting our attention to n -skeleta, the following short exact sequence of chain complexes of $\mathbb{Z}H$ -modules arises

$$(4.3) \quad 0 \rightarrow C_* (\tilde{X}^{(n)}) \rightarrow C_* (\tilde{Y}^{(n)}) \rightarrow C_* (\tilde{Y}^{(n)}, \tilde{X}^{(n)}) \rightarrow 0.$$

Consider the induced long exact homology sequence

$$(4.4) \quad 0 \rightarrow \tilde{H}_n (\tilde{X}^{(n)}) \rightarrow \tilde{H}_n (\tilde{Y}^{(n)}) \rightarrow \tilde{H}_n (\tilde{Y}^{(n)}, \tilde{X}^{(n)}) \rightarrow \tilde{H}_{n-1} (\tilde{X}^{(n)}) \rightarrow \dots$$

Since X is the $(n+1)$ -skeleton of an $K(H, 1)$, the homology group $\tilde{H}_{n-1}(\tilde{X}^{(n)})$ is trivial. Now the exact sequence (4.4) can be truncated, obtaining the short exact sequence

$$(4.5) \quad 0 \rightarrow Z_n (\tilde{X}) \xrightarrow{\iota} Z_n (\tilde{Y}) \rightarrow Z_n (\tilde{Y}, \tilde{X}) \rightarrow 0,$$

where ι is induced by the inclusion $\tilde{X} \subseteq \tilde{Y}$. We claim that the short exact sequence (4.5) satisfies the three hypothesis of Lemma 2.10.

First, since X is a finite cell complex, $C_{n+1}(\tilde{X})$ is finitely generated as a $\mathbb{Z}H$ -module. Therefore $Z_n(\tilde{X})$ is also finitely generated as a $\mathbb{Z}H$ -module.

Second, the construction of Y guarantees that $Z_n(\tilde{Y})$ is a free $\mathbb{Z}G$ -module, hence $Z_n(\tilde{Y})$ is a free $\mathbb{Z}H$ -module.

Third, we need to verify that $Z_n(\tilde{Y}, \tilde{X})$ is a projective $\mathbb{Z}H$ -module; in fact we show that it is stably free. Indeed, since $X^{(n)}$ and $Y^{(n)}$ are the $(n+1)$ -skeleta of a $K(H, 1)$ and a $K(G, 1)$ respectively, the reduced homology groups $\tilde{H}_k(\tilde{X}^{(n)})$ and $\tilde{H}_k(\tilde{Y}^{(n)})$ are trivial for $1 \leq k < n$. Then, considering the exact sequence (4.4), we have that

$$(4.6) \quad 0 \rightarrow Z_n (\tilde{Y}^{(n)}, \tilde{X}^{(n)}) \rightarrow C_n (\tilde{Y}^{(n)}, \tilde{X}^{(n)}) \rightarrow \dots \rightarrow C_0 (\tilde{Y}^{(n)}, \tilde{X}^{(n)}) \rightarrow 0$$

is also exact. Since all the $\mathbb{Z}H$ -modules $C_i(\tilde{Y}^{(n)}, \tilde{X}^{(n)})$ are free, and application of Schanuel's Lemma to (4.6) and a trivial resolution of $C_0(\tilde{Y}^{(n)}, \tilde{X}^{(n)})$ shows that $Z_n(\tilde{Y}^{(n)}, \tilde{X}^{(n)})$ is a stably free $\mathbb{Z}H$ -module.

Thus we have shown that the short exact sequence (4.5) satisfies the three hypothesis of Lemma 2.10. Before invoking this lemma and concluding the proof, we set up notation for the norms required to specify representatives of FV_G^{n+1} and FV_H^{n+1} .

Let $\|\cdot\|_1$ denote the ℓ_1 -norm on $C_i(\tilde{Y})$ induced by the basis consisting on all i -cells of \tilde{Y} . Let $\|\cdot\|_{Z_n(\tilde{Y})}$ denote the ℓ_1 -norm on $Z_n(\tilde{Y})$ induced by a free $\mathbb{Z}G$ -basis; by definition this is also filling norm on $Z_n(\tilde{Y})$. Then (a representative of) FV_G^{n+1} is given by

$$(4.7) \quad FV_G^{n+1}(k) = \max \{ \|\gamma\|_{Z_n(\tilde{Y})} : \gamma \in Z_1(\tilde{Y}), \|\gamma\|_1 \leq k \}.$$

Since $C_{n+1}(\tilde{X}) \subseteq C_{n+1}(\tilde{Y})$ is a free factor, the ℓ_1 -norm on $C_{n+1}(\tilde{X})$ induced by the $(n+1)$ -cells of \tilde{X} equals the restriction of $\|\cdot\|_1$ to $C_{n+1}(\tilde{X})$. Let $\|\cdot\|_{Z_n(\tilde{X})}$ denote the filling-norm on $Z_n(\tilde{X})$ as a $\mathbb{Z}H$ -module induced by the boundary map $C_{n+1}(\tilde{X}) \xrightarrow{\partial_{n+1}} Z_n(\tilde{X})$. Then

$$(4.8) \quad FV_H^{n+1}(k) = \max \{ \|\gamma\|_{Z_n(\tilde{X})} : \gamma \in Z_1(\tilde{X}), \|\gamma\|_1 \leq k \}.$$

By Lemma 2.10 applied to the short exact sequence (4.5), there exists a constant $C_1 > 0$ and a morphism of $\mathbb{Z}H$ -modules $\rho: Z_n(\tilde{Y}) \rightarrow Z_n(\tilde{X})$ such that

$$(4.9) \quad \|\rho(\alpha)\|_{Z_n(\tilde{X})} \leq C_1 \cdot \|\alpha\|_{Z_n(\tilde{Y})},$$

for every $\alpha \in Z_n(\tilde{Y})$, and $\rho \circ \iota$ is the identity on $Z_n(\tilde{X})$.

Let $k \in \mathbb{N}$ and let $\gamma \in Z_n(\tilde{X})$ such that $\|\gamma\|_1 \leq k$. Then (4.9) implies that

$$(4.10) \quad \|\gamma\|_{Z_n(\tilde{X})} = \|\rho \circ \iota(\gamma)\|_{Z_n(\tilde{X})} \leq C \cdot \|\iota(\gamma)\|_{Z_n(\tilde{Y})} \leq C \cdot FV_G^{n+1}(k).$$

Since γ was arbitrary, we have $FV_H^{n+1}(k) \leq C \cdot FV_G^{n+1}(k)$. \square

Remark 4.5. *The proof of Theorem 4.4 does not apply to obtain that $FV_H^{m+1} \leq FV_G^{m+1}$ for $m < n$. As mentioned in the introduction, that statement is false. The argument breaks down since $Z_m(\widetilde{M}_f)$ is not projective if $m < n$.*

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